Engineering Notes

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Finite-Time Matrix Convolution Integral Sensitivity Calculations

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Introduction

In the process of optimally adjusting the control weighting matrices, structural parameters, sensor locations, and actuator locations for finite-time optimal tracking, terminal tracking, and disturbance-accommodating tracking problems,[1] the analyst must compute the partial derivatives of matrix convolution integrals. Even though analytic representations of the sensitivity partials can be obtained, the expressions are complicated to deal with and second- and higher-order partials are cumbersome to obtain. Alternatively, we present in this Note a straightforward algorithm for computing the partial derivatives. The method is based on an augmented matrix approach whereby a single-matrix exponential solution provides the desired partial derivatives.

Convolution Integral Partial Derivatives

A general representation for a convolution matrix integral is given by

$$G(t) = \int_0^t e^{A(t-r)} B e^{Cr}\, \mathrm{d}r \tag{1}$$

where the constant matrices $A(n_1 \times n_1)$, $B(n_1 \times n_2)$, and $C(n_2 \times n_2)$ are specified. In this Note we assume that $\boldsymbol{p}$ is a vector of free parameters that are to be adjusted in an optimization process, where

$$A = A(\boldsymbol{p}) \qquad B = B(\boldsymbol{p}) \qquad C = C(\boldsymbol{p})$$

From elementary calculus, the partial derivative of Eq. (1) with respect to the ith design variable follows as

$$\frac{\partial G(t)}{\partial p_i} = \int_0^t \left[\frac{\partial e^{A(t-r)}}{\partial p_i} B e^{Cr} + e^{A(t-r)} \frac{\partial B}{\partial p_i} e^{Cr} + e^{A(t-r)} B \frac{\partial e^{Cr}}{\partial p_i} \right] \mathrm{d}r \tag{2}$$

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From Ref. 2, Eq. (2) can be written as

$$\frac{\partial G(t)}{\partial p_i} = \int_0^t \int_0^{(t-r)} e^{A(t-r-s)} \frac{\partial A}{\partial p_i} e^{As} B e^{Cr}\, \mathrm{d}s\, \mathrm{d}r + \int_0^t e^{A(t-r)} \frac{\partial B}{\partial p_i} e^{Cr}\, \mathrm{d}r + \int_0^t \int_0^r e^{A(t-r)} B e^{C(r-s)} \frac{\partial C}{\partial p_i} e^{Cs}\, \mathrm{d}s\, \mathrm{d}r \tag{3}$$

To evaluate Eq. (3), the first double-matrix convolution integral is transformed to standard form. This is accomplished by introducing the coordinates u and v defined by the equations[3]

$$u = r, \qquad v = r + s, \qquad \mathrm{d}s\, \mathrm{d}r = \left| \frac{\partial(r,s)}{\partial(u,v)} \right| \mathrm{d}u\, \mathrm{d}v \tag{4}$$

By suitably adjusting the limits of integration in the transformed region of the (u,v) plane, the first integral in Eq. (3) can be written as

$$\int_0^t \int_0^v e^{A(t-v)} \frac{\partial A}{\partial p_i} e^{A(v-u)} B e^{Cu}\, \mathrm{d}u\, \mathrm{d}v \tag{5}$$

Introducing Eq. (5) into Eq. (3), the three integrals can be evaluated by defining the following three constant matrices and computing the associated matrix exponentials:[4]

$$v_1 = \begin{bmatrix} A & \partial A/\partial p_i & 0 \\ 0 & A & B \\ 0 & 0 & C \end{bmatrix} \qquad e^{v_1 t} = \begin{bmatrix} F_{11}(t) & G_{11}(t) & H_{11}(t) \\ 0 & F_{21}(t) & G_{21}(t) \\ 0 & 0 & F_{31}(t) \end{bmatrix}$$

$$v_2 = \begin{bmatrix} A & \partial B/\partial p_i \\ 0 & C \end{bmatrix} \qquad e^{v_2 t} = \begin{bmatrix} F_{12}(t) & G_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix}$$

$$v_3 = \begin{bmatrix} A & B & 0 \\ 0 & C & \partial C/\partial p_i \\ 0 & 0 & C \end{bmatrix} \qquad e^{v_3 t} = \begin{bmatrix} F_{13}(t) & G_{13}(t) & H_{13}(t) \\ 0 & F_{23}(t) & G_{23}(t) \\ 0 & 0 & F_{33}(t) \end{bmatrix}$$

leading to

$$\frac{\partial G(t)}{\partial p_i} = H_{11}(t) + G_{12}(t) + H_{13}(t)$$

The presented solution is particularly useful if any of the partial derivatives of A, B, or C vanish; however, the extension to higher-order partial derivatives is cumbersome.

Augmented Matrix Technique

In this section we present a solution technique that computes the three integrals appearing in Eq. (3) with a single-matrix exponential solution and generalizes the method for higher-order partial derivatives. It is well-known that the solution for $G(t)$ in Eq. (1) is obtained by integrating the system equations

$$\dot{X}(t) = DX(t) \qquad (n_1 + n_2) \times n_2 \tag{6}$$

leading to

$$X(t) = e^{Dt} X(0) \tag{7}$$

where

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} G(t) \\ e^{Ct} \end{bmatrix} \qquad X(0) = \begin{bmatrix} 0 \\ I \end{bmatrix} \qquad D = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{matrix} \}n_1 \\ \}n_2 \end{matrix}$$

First-Order Partial Derivatives

To obtain the partial derivative of $G(t)$, we partially differentiate Eq. (6) with respect to the ith design variable p_i, leading to

$$\frac{\partial \dot{X}(t)}{\partial p_i} = \frac{\partial D}{\partial p_i} X(t) + D \frac{\partial X(t)}{\partial p_i}; \qquad \frac{\partial X(0)}{\partial p_i} = [0,0]^T \tag{8}$$

By combining Eqs. (6) and (8), we obtain the augmented system

$$\dot{Y}_i(t) = F_i Y_i(t) \qquad (i = 1,...,m) \tag{9}$$

leading to

$$Y_i(t) = e^{F_i t} Y_i(0) \tag{10}$$

where

$$Y_i(t) = \left[\frac{\partial X(t)}{\partial p_i}, X(t)\right]^T; \qquad Y_i(0) = \left[\frac{\partial X(0)}{\partial p_i}, X(0)\right]^T$$

$$F_i = \begin{bmatrix} D & \partial D/\partial p_i \\ 0 & D \end{bmatrix}\begin{matrix} \}\mu \\ \}\mu \end{matrix}, \quad \mu = n_1 + n_2$$

Defining the 16 partitions of $e^{F_i t}$ as

$$\varepsilon^{F_i T} = \begin{bmatrix} \phi_{11}(t) & \cdots & \phi_{14}(t) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \phi_{41}(t) & \cdots & \phi_{44}(t) \end{bmatrix}_i$$

it then follows that the partial derivatives are obtained from the equation

$$Y_i(t) = [\phi_{14}(t), \phi_{24}(t), \phi_{34}(t), \phi_{44}(t)]_i^T = \left[\frac{\partial G(t)}{\partial p_i}, \frac{\partial X_2(t)}{\partial p_i}, G(t), X_2(t)\right]^T \tag{11}$$

Second-Order Partial Derivatives

To obtain second-order partial derivatives of $G(t)$, we partially differentiate Eq. (9) with respect to the jth design variable p_j, leading to

$$\frac{\partial \dot{Y}_i(t)}{\partial p_j} = \frac{\partial F_i}{\partial p_j} Y_i(t) + F_i \frac{\partial Y_i(t)}{\partial p_j}; \qquad \frac{\partial Y_i(0)}{\partial p_j} = [0,0]^T \tag{12}$$

By combining Eqs. (9) and (12), we obtain the augmented system

$$\dot{Z}_{ij}(t) = H_{ij} Z_{ij}(t); \qquad (i,j = 1,2,...,m) \tag{13}$$

leading to

$$Z_{ij}(t) = e^{H_{ij}(t)} Z_{ij}(0) \tag{14}$$

where

$$Z_{ij}(t) = \left[\frac{\partial Y_i(t)}{\partial p_j}, Y(t)\right]^T; \; Z_{ij}(0) = \left[\frac{\partial Y_i(0)}{\partial p_j}, Y_i(0)\right]^T \Big/ (4\mu \times n_2)$$

$$H_{ij} = \begin{bmatrix} F_i & \partial F_i/\partial p_j \\ 0 & F_i \end{bmatrix}\begin{matrix} \}2\mu \\ \}2\mu \end{matrix}$$

Defining the 64 partitions of $e^{H_{ij} t}$ as

$$e^{H_{ij} t} = \begin{bmatrix} \phi_{11}(t) & \cdots & \phi_{18}(t) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \phi_{81}(t) & \cdots & \phi_{88}(t) \end{bmatrix}_{ij}$$

it then follows that the sensitivity partials are obtained from the equation

$$Z_{ij}(t) = [\phi_{18}(t), \phi_{28}(t), \phi_{38}(t), \phi_{48}(t), \phi_{58}(t), \phi_{68}(t), \phi_{78}(t), \phi_{88}(t)]_{ij}^T = \left[\frac{\partial^2 G(t)}{\partial p_i \partial p_j}, \frac{\partial^2 X_2(t)}{\partial p_i \partial p_j}, \frac{\partial G(t)}{\partial p_j}, \frac{\partial X_2(t)}{\partial p_j}, \frac{\partial G(t)}{\partial p_i}, \frac{\partial X_2(t)}{\partial p_i}, G(t), X_2(t)\right]^T \tag{15}$$

Extensions to higher-order partial derivatives follow a similar pattern.

Example Application

The A, B, and C matrices of Eq. (1) and their associated first- and second-order partial derivatives are assumed to be given by

$$A = \begin{bmatrix} p_1 p_2 & p_1 p_2^2 \\ p_1^2 p_2 & p_2^2 \end{bmatrix}, \quad \frac{\partial A}{\partial p_1} = \begin{bmatrix} p_2 & p_2^2 \\ 2p_1 p_2 & 0 \end{bmatrix},$$

$$\frac{\partial A}{\partial p_2} = \begin{bmatrix} p_1 & 2p_1 p_2 \\ p_1^2 & 2p_2 \end{bmatrix}, \quad \frac{\partial^2 A}{\partial p_1 \partial p_2} = \begin{bmatrix} 1 & 2p_2 \\ 2p_1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} p_1 + p_2 & p_1^2 + p_2 \\ p_1 + p_2^3 & p_1^3 \end{bmatrix}, \quad \frac{\partial B}{\partial p_1} = \begin{bmatrix} 1 & 2p_1 \\ 1 & 3p_1^2 \end{bmatrix},$$

$$\frac{\partial B}{\partial p_2} = \begin{bmatrix} 1 & 1 \\ 3p_2^2 & 0 \end{bmatrix}, \quad \frac{\partial^2 B}{\partial p_1 \partial p_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} p_1^2 p_2 & p_2 \\ p_1 & p_1 p_2 \end{bmatrix}, \quad \frac{\partial C}{\partial p_1} = \begin{bmatrix} 2p_1 p_2 & 0 \\ 1 & p_2 \end{bmatrix},$$

$$\frac{\partial C}{\partial p_2} = \begin{bmatrix} p_1^2 & 1 \\ 0 & p_1 \end{bmatrix}, \quad \frac{\partial^2 C}{\partial p_1 \partial p_2} = \begin{bmatrix} 2p_1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\{p_1, p_2\} = \{2,3\}$$

By introducing these matrices into the D matrix of Eq. (6), the F_1 matrix of Eq. (9), and the H_{ij} matrix of Eq. (13), we find from Eq. (15) that

$$\frac{\partial^2 G(t)}{\partial p_1 \partial p_2} = \begin{bmatrix} 14.68 & 5.950 \\ 13.25 & 5.370 \end{bmatrix} \times 10^{12} \quad \frac{\partial G(t)}{\partial p_2} = \begin{bmatrix} 11.59 & 4.654 \\ 10.40 & 4.176 \end{bmatrix} \times 10^{10}$$

$$\frac{\partial G(t)}{\partial p_1} = \begin{bmatrix} 12.13 & 5.105 \\ 11.03 & 4.643 \end{bmatrix} \times 10^{10} \qquad G(t) = \begin{bmatrix} 9.940 & 4.154 \\ 8.987 & 3.755 \end{bmatrix} \times 10^{10}$$

These numerical values have been verified by using finite-difference techniques for calculating numerical partial derivatives.

Conclusions

A straightforward algorithm has been presented for generating nth-order partial derivatives of a standard matrix convolution integral. The solution technique is based on defining an augmented matrix. A numerical example is provided which demonstrates the basic method for computing first- and second-order partial derivatives.

References

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Low-Thrust Insertion into Orbit around Mars

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Introduction

THE Mars Observer spacecraft is targeted for a circular mapping orbit of about 350-km altitude above Mars (radius = 3747.2 km) with an inclination of 98.8 deg and a 2 a.m. ascending node (sun synchronous). This orbit is achieved in two steps: 1) capture in an elliptical orbit around Mars and 2) subsequent circularization at the mapping orbit altitude. The initial elliptical orbit is also the phasing orbit for obtaining the correct ascending node position.

Theoretically, insertion of a spacecraft from its approach trajectory into the elliptical capture orbit around Mars can be accomplished by an impulsive burn at the common periapsis of the approach hyperbola and the capture ellipse. However, the Mars Observer spacecraft requires partial deployment of its appendages and solar array in the interplanetary cruise mode. The limited structural strength of these appendages requires restowage for high-thrust maneuvers. Rather than provide for restowing with attendant weight and reliability penalties, low-thrust insertion is proposed using four 110-lbf bipropellant thrusters, providing acceleration levels less than 0.15 g during the insertion burn.

Selection of the Elliptical Capture (Phasing) Orbit

For a launch between Aug. 20 and Sept. 9, 1990, and an arrival between Aug. 12 and Aug. 30, 1991, the capture orbit around Mars will have an ascending node at approximately 4 a.m.[1] A 30-deg node shift is then necessary in the phasing orbit to attain the required 2 a.m. node. Thereafter, a sequence of maneuvers is required to attain the mapping orbit. These maneuvers must be completed before the solar conjunction period between Oct. 24 and Nov. 24, 1991. Therefore, the primary consideration in choosing the phasing orbit is the attainment of the correct nodal rate.

For an impulsive burn, the target phasing orbit could be a circular orbit of 350-km altitude, but with an inclination of 90 deg. After achieving the correct nodal position, a subsequent burn could be effected to change the inclination to the required 98.8 deg.

With the 440-lbf thrust available on the Mars Observer spacecraft, capture into this 350-km phasing orbit requires a burn on the order of 30 min. This would contribute a finite-burn penalty of over 100 m/s because of the long arc. This penalty is the additional velocity increment required over an impulsive burn, resulting from the low thrust level that is accompanied by a relatively long thrust arc. To minimize this penalty, the plan is to inject into an elliptical capture orbit that, even with the same inclination as the final mapping orbit, provides the desired nodal shift. Table 1 shows the chosen nominal capture orbit which, for a launch on the first day of the window, accomplishes the required node shift in about 65 days. The target periapsis radius of this capture orbit is higher than the nominal 350 km to assure that the spacecraft stays above 300 km with 99% confidence in the presence of the navigational error (Table 2).[1] The northern approach to Mars has an advantage over the southern approach in that the resulting orbit has an ascending node 4 deg closer to that of the mapping orbit.

Approach Hyperbola and Burn Initiation

After the final trajectory correction maneuver (TCM 4 nominally 10 days prior to arrival at Mars), the spacecraft is targeted to the aim point (point B in Fig. 1) in the B-plane on the incoming leg of a hyperbolic trajectory. Because of the low thrust level (440 lbf), the insertion burn is estimated to last about 15 min, covering an arc of about 40 deg (Fig. 2). The figure also shows that during the burn, the periapsis of the orbit decreases. A burn over an arc of 40 deg results in a decrease of

Table 1 Nominal capture orbit for launch day 1

Parameter	Value
Semimajor axis	20,000 km
Periapsis radius	4,100 km
Eccentricity	0.795
Inclination	92.78 deg
Argument of perigee	92.63 deg

Table 2 Approach navigation uncertainty

Parameter	3-sigma value
Semimajor axis	330 km
Semiminor axis	180 km
Orientation angle	60 deg
Linearized time of flight	120 s

Table 3 Reduction in periapsis during insertion burn (using all four thrusters)

Percent of burn completed	Periapsis radius (km)
0	4150
25	4135
50	4128
75	4120
100	4100

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